Critical two-point function for long-range self-avoiding walks with power-law couplings

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Join work with Akira Sakai

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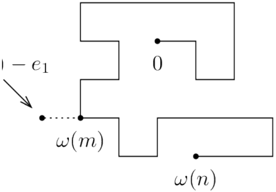
Self-avoiding walk



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II. Self-avoiding Walk (SAW)

the two point function of SAW on $\mathbb{Z}^d$ from the origin o to x by $\varphi_0(x) = \delta_{o,x}$ and, for $n \geq 1$,

$$\varphi_n(x) = \sum_{w \in \mathcal{W}_n(x)} \prod_{i=1}^{n} D(w_i - w_{i-1}) \prod_{0 \leq i < j \leq n} (1 - \delta_{w_i, w_j}).$$

The primary models in this talk are defined as follows:

1. Finite-range model

(a) Nearest-neighbor model:

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(A)

$$D(x) = \frac{|x/L|^{-d-\alpha}}{\sum_{y \in \mathbb{Z}^d} |y/L|^{-d-\alpha}}.$$

(B) Let $\zeta(s) = \sum_{t=1}^{\infty} t^{-s}$ be the Riemann-zeta function and

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Next, let $h \geq 0$ be bounded function, piecewise continuous, symmetric and supported in $[-1,1]^d$. then, for large L, we define

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Combining these distributions, we define the compound-zeta distribution as follows

$$D(x) = \sum_{t \in \mathbb{N}} U_L^{*t}(x)\, T_\alpha(t).$$

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For finite-range model, it is believe that there are critical exponents γ, ν and η such that

$$G_{p_c}(x) \approx \frac{1}{|x|^{d-2+\eta}}, \quad \chi_p \approx_{p \uparrow p_c} (p_c - p)^{-\gamma}, \quad \xi^{(2)}(n) \approx n^{\nu}.$$

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Note that forfinite-range model $d_c = 4$ and for $d > 3$

$$G_{p_c}^{RW}(x) \approx \frac{1}{|x|^{d-2}}, \quad \chi_p^{RW} \approx_{p \uparrow p_c} (p_c - p)^{-1}, \quad \xi^{RW,(2)}(n) \approx n^{\frac{1}{2}}.$$

Theorem (Aizemann and Newmann (1984))
Suppose

$$\sup_{x\in\mathbb{Z}} G^{*2}_{p_c}(x) := \sum_{y\in\mathbb{Z}^d} G_{p_c}(x)G_{p_c}(x-y) < \infty,$$

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$$\xi^{(r)}(n) \approx \begin{cases} n^{\frac{1}{\alpha \wedge 2}} & (\alpha \neq 2), \\ (n \log \sqrt{n})^{\frac{1}{2}} & (\alpha = 2). \end{cases} \qquad \left(\nu = \frac{1}{\alpha \wedge 2} \text{ for } \alpha \neq 2\right)$$

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Modify Fisher's relation for long-range model with $\alpha \neq 2$ as follows:

$$\gamma = (\alpha \wedge 2 - \eta)\nu.$$

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Theorem (Hofstad, Hara, Slade (2003), Hara (2009)) For nearest-neighbor model and $d > 4$, then

$$G_{p_c}(x) \sim_{|x|\to\infty} \frac{a_d}{p_c \sigma^2 |x|^{d-2}},$$

where

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Assumptions

$$(1)\, D(x) \approx (|x| \vee L)^{-d-\alpha} \quad \text{for } \alpha > 0,$$

$$(2)\, 1 - \hat{D}(k) = v_\alpha |k|^{\alpha\wedge 2} \times \begin{cases} 1 + O\big((L|k|)^\epsilon\big) & (\alpha \neq 2), \\ \log \frac{1}{L|k|} + O(1) & (\alpha = 2), \end{cases}$$

$$(3)\ \|D^{*n}\|_\infty \le \begin{cases} O(L^{-d})n^{-\frac{d}{\alpha\wedge 2}}, & \alpha \neq 2, \\ O(L^{-d})(n\log n)^{-\frac{d}{2}}, & \alpha = 2, \end{cases}$$

$$1-\hat{D}(k) \begin{cases} < 1-\Delta, & k \in [-\pi,\pi]^d, \\ > \Delta, & \|k\|_\infty \ge L^{-1}, \end{cases} \quad \Delta \in (0,1),$$

$$(4)\ D^{*n}(x) \le \frac{O(L^{\alpha\wedge 2})n}{(|x|\vee L)^{d+\alpha\wedge 2}} \times \begin{cases} 1, & \alpha \neq 2, \\ \log(|x|\vee L), & \alpha = 2, \end{cases}$$

$$(5)\ \left| D^{*n}(x) - \frac{D^{*n}(x+y)+D^{*n}(x-y)}{2} \right|$$

$$\le \frac{O(1)(|y|\vee L)^{\alpha\wedge 2}n}{(|x|\vee L)^{d+\alpha\wedge 2+2}} \times \begin{cases} 1, & \alpha \neq 2, \\ \log(|x|\vee L), & \alpha = 2 \end{cases}$$

for all $|y| \le \frac{1}{3}|x|$.

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Theorem (Chen and Sakai, 2015) Under the assumption (1)-(5) on D, for $\alpha \neq 2$, $L \gg$ and $d > 2(\alpha \wedge 2)$, we have

$$G_{p_c}(x) \sim_{|x| \to \infty} \frac{\gamma_2}{p_c v_\alpha |x|^{d-\alpha\wedge 2}},$$

where

$$\gamma_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}.$$

Corollary Under the assumption (1)-(4) on D, for long-range spread-out RW with $\alpha \neq 2$ and any $L \geq 1$ and $d > (\alpha \wedge 2)$, we obtain

$$G_1^{RW}(x) \sim_{|x| \to \infty} \frac{\gamma_\alpha}{v_\alpha |x|^{d-\alpha\wedge 2}},$$

where

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Theorem (Chen and Sakai, 201?) For long-range spread-out model (B) with $\alpha = 2$, $L \gg$ and $d \geq 4$, then

$$G_{p_c}(x) \sim_{|x|\to\infty} \frac{\Gamma(\frac{d}{2}-1)}{4p_c\pi^{\frac{d}{2}} v_2 |x|^{d-2} \log|x|}.$$

Corollary Under the assumption (1)-(4) on D with $\alpha = 2$, $L \gg$ and $d > 2$, we obtain

$$G_1^{RW}(x) \sim_{|x|\to\infty} \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2} v_2 |x|^{d-2} \log|x|},$$

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Using the inverse formula

$$G_1^{RW}(x) = \lim_{p\uparrow 1} G_s(x) = \lim_{p\uparrow 1} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \hat{G}_p^{RW}(k) e^{-ik\cdot x},$$

and

$$\hat{G}_p^{RW}(k) = \sum_{n=0}^{\infty} \hat{D}(k)^n p^n = \frac{1}{1 - \hat{D}(k)p}.$$

By monotone convergence theorem, we obtain

$$G_1^{RW}(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{e^{-ik\cdot x}}{1 - \hat{D}(k)}.$$

Using the following formula

$$\frac{1}{r^{\alpha\wedge 2}} = \frac{1}{\Gamma(\frac{\alpha\wedge 2}{2})}\int_0^\infty t^{\frac{\alpha\wedge 2}{2}-1} e^{-r^2 t}\,\mathrm{d}t \quad \text{for any } r > 0,$$

with $r = (1-\hat{D}(k))^{\frac{1}{\alpha\wedge 2}}$, for any $d > \alpha \wedge 2$ we have

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By Assumption (4):

$$D^{*n}(x) \approx \frac{L^{\alpha\wedge 2} n}{|x|^{d+\alpha\wedge 2}} \times \begin{cases} 1, & \alpha \neq 2, \\ \log|x|, & \alpha = 2, \end{cases}$$

we can obtain

$$G_1^{RW}(x) \sim \frac{1}{\Gamma(\frac{\alpha\wedge 2}{2})} \int_0^\infty \mathrm{d}t\ t^{\frac{\alpha\wedge 2}{2}-1} \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-(1-\hat{D}(k))^{\frac{2}{\alpha\wedge 2}} t - ik\cdot x}.$$

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By Assumption (2) $\exists v_\alpha > 0$ such that

$$1-\hat{D}(k) = v_\alpha |k|^{\alpha\wedge 2}\times\begin{cases}1+O\big((L|k|)^\epsilon\big) & (\alpha\neq 2),\\ \log\frac{1}{L|k|}+O(1) & (\alpha=2),\end{cases}$$

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we have

$$G_1^{RW}(x) \sim_{|x|\uparrow\infty} \begin{cases}\frac{1}{\Gamma(\frac{\alpha\wedge 2}{2})}\int_0^\infty \mathrm{d}t\; t^{\frac{\alpha\wedge 2}{2}-1}\int_{\mathbb{R}^d}\frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-tv_\alpha^{\frac{2}{\alpha\wedge 2}}|k|^2-ik\cdot x}, & (\alpha\neq 2),\\ \int_0^\infty \mathrm{d}t\; \int_{\mathbb{R}^d}\frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-tv_\alpha|k|^2\log(\frac{1}{L|k|})-ik\cdot x}, & (\alpha=2).\end{cases}$$

Therefore

$$G_1^{RW}(x) \sim \begin{cases} \frac{\gamma_\alpha}{v_\alpha |x|^{d-\alpha\wedge 2}}, & (\alpha \neq 2) \\ \frac{\gamma_\alpha}{v_2 |x|^{d-2} \log |x|}, & (\alpha = 2), \end{cases} \quad \text{where } \gamma_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}.$$

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In particular, for $\alpha = 2$ case

$$\sup_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} G_{p_c}(x-y)G_{p_c}(y) \lesssim \sum_{y\in\mathbb{Z}^d}\Big(\frac{c}{|x|^{d-2}\log|x|}\Big)^2 < \infty \quad \text{for } d \geq 4.$$

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For mathematical proof, we need "lace expansion" and "bootstrapping argument". To use lace expansion we need assumption (5) as follows

$$(5) \qquad \left| D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \right| \leq \frac{(|y| \vee L)^{\alpha\wedge 2} n}{(|x| \vee L)^{d+\alpha\wedge 2+2}} \times \begin{cases} 1, & \alpha \neq 2, \\ \log(|x| \vee L), & \alpha = 2. \end{cases}$$

However "bootstrapping argument" doesn't work using assumption (4) and (5) for all dimensions if $\alpha = 2$. We need sharper estimate by model (B) for $\alpha = 2$ as follows:

$$(4')\ D^{*n}(x) \le O(1)\Big(\frac{1}{L^2 n \log(|x| \vee L)}\Big)^{\frac{d}{2}} e^{-\frac{(|x|\vee L)^2}{2L^2 n \log(|x|\vee L)}},$$

$$(5')\ \left| D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \right|$$

$$\le O(1)(|y| \vee L)^2 \Big(\frac{1}{L^2 n \log(|x| \vee L)}\Big)^{\frac{d}{2}+1} e^{-\frac{(|x|\vee L)^2}{2L^2 n \log(|x|\vee L)}}$$

for all $|y| \le \frac{1}{3}|x|$.

Thank You